# Proof of "Real-Time Stochastic Home Energy Management with HVAC Load and Energy Storage"

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## I. Related Equations

The objective function is

$$\begin{aligned}
&\mathcal{G}_h^*(\boldsymbol{s}_h;\boldsymbol{z}_h) = \\
&\min_{\boldsymbol{x}_h}\ \mathcal{G}_h(\boldsymbol{x}_h,\boldsymbol{s}_h;\boldsymbol{z}_h) = \mathcal{F}_h(\boldsymbol{x}_h;\boldsymbol{z}_h) + \beta\mathbb{E}[\mathcal{G}_{h+1}^*(\boldsymbol{s}_{h+1};\boldsymbol{z}_{h+1})] \\
&\text{s.t. } T_{h+1}^{in} = \epsilon T_h^{in} + (1-\epsilon)\Big(T_h^{out} + \frac{\eta}{A}e_h\Big), && \text{(1a)}\\
&\quad \underline{T} \le T_{h+1}^{in} \le \overline{T}, && \text{(1b)}\\
&\quad 0 \le e_h \le \bar{e}, && \text{(1c)}\\
&\quad 0 \le \Delta d_h \le \Delta\bar{d}_h, && \text{(1d)}\\
&\quad E_{h+1} = E_h + \begin{cases}\eta_1 p_h, & p_h > 0\\ \frac{1}{\eta_2}p_h, & p_h \le 0\end{cases}, && \text{(1e)}\\
&\quad 0 \le E_{h+1} \le \bar{E}, && \text{(1f)}\\
&\quad \eta_1 p_h \le \bar{p}, -\frac{1}{\eta_2}p_h \le \bar{p}. && \text{(1g)}\\
&\quad 0 \le \Delta d_h \le \Delta\bar{d}_h, && \text{(1h)}\\
&\quad \tau_h = d_h - \Delta d_h + e_h + p_h - g_h - r_h, && \text{(1i)}\\
&\quad \underline{\tau}_h \le \tau_h \le \bar{\tau}_h, && \text{(1j)}
\end{aligned}$$

We simplify (1) by replacing the expected future cost $G_{h+1}^*(\boldsymbol{s}_h;\boldsymbol{z}_h)$ by another term $H_{h+1}^*(\boldsymbol{s}_h;\boldsymbol{z}_h)$, so that the problem becomes

$$\begin{aligned}
&\hat{\mathcal{G}}_h^*(\boldsymbol{s}_h;\boldsymbol{z}_h) = \\
&\min_{\boldsymbol{x}_h}\ \hat{\mathcal{G}}_h(\boldsymbol{x}_h,\boldsymbol{s}_h;\boldsymbol{z}_h) = \mathcal{F}_h(\boldsymbol{x}_h;\boldsymbol{z}_h) + \beta\mathbb{E}[\mathcal{H}_{h+1}^*(\boldsymbol{s}_{h+1};\boldsymbol{z}_{h+1})] \\
&\text{s.t. } (1a) - (1j).
\end{aligned} \tag{2}$$

In particular, $\mathcal{H}_{h+1}^*$ differs from $G_{h+1}^*$ in the sense that the constraint (1j) is omitted:

$$\begin{aligned}
&\mathcal{H}_{h+1}^*(\boldsymbol{s}_{h+1};\boldsymbol{z}_{h+1}) = \\
&\min_{\boldsymbol{x}_{h+1}}\ \mathcal{F}_h(\boldsymbol{x}_{h+1};\boldsymbol{z}_{h+1}) + \beta\mathbb{E}[\mathcal{H}_{h+2}^*(\boldsymbol{s}_{h+2};\boldsymbol{z}_{h+2})] \\
&\text{s.t. } (1a) - (1i).\text{[1]}
\end{aligned} \tag{3}$$

## Appendix A
## Proof of Lemma 1

*Proof.* To prove that $\mathcal{H}_h^*(\boldsymbol{s}_h;\boldsymbol{z}_h)$ is convex in $\boldsymbol{s}_h$, we will show that for any $\boldsymbol{s}_h^1 = \{T_h^{in,1}, E_h^1\}$ and $\boldsymbol{s}_h^2 = \{T_h^{in,2}, E_h^2\}$,

$$\alpha\mathcal{H}_h^*(\boldsymbol{s}_h^1;\boldsymbol{z}_h) + (1-\alpha)\mathcal{H}_h^*(\boldsymbol{s}_h^2;\boldsymbol{z}_h) \ge \mathcal{H}_h^*(\boldsymbol{s}_h^3;\boldsymbol{z}_h), \tag{4}$$

where $\alpha \in [0,1]$ and $\boldsymbol{s}_h^3 = \{\alpha T_h^{in,1} + (1-\alpha)T_h^{in,2}, \alpha E_h^1 + (1-\alpha)E_h^2\}$.

Let $[\hat{\boldsymbol{x}}_h^1, \hat{\boldsymbol{x}}_{h+1}^1, \ldots], [\hat{\boldsymbol{x}}_h^2, \hat{\boldsymbol{x}}_{h+1}^2, \ldots]$ denote the sequence of decisions following from the states $\boldsymbol{s}_h^1$ and $\boldsymbol{s}_h^2$, respectively. Following the state $\boldsymbol{s}_h^3$, a heuristic way $\pi_{heu}$ is to produce a decision as a convex combination of $\hat{\boldsymbol{x}}_h^1$ and $\hat{\boldsymbol{x}}_h^2$.

$$\pi_{heu}(\boldsymbol{s}_h^3, \boldsymbol{z}_h) = \alpha\hat{\boldsymbol{x}}_h^1 + (1-\alpha)\hat{\boldsymbol{x}}_h^2, \forall h. \tag{5}$$

As all constraints in Problem (3) are convex, the heuristic way $\pi_{heu}$ is feasible for Problem (3). According to the definition of optimality, the heuristic way achieves no less operating cost than the optimal function $\pi_2$, i.e.,

$$\mathcal{O}_h(\pi_{heu}, \boldsymbol{s}_h^3, \boldsymbol{z}_h) \ge \mathcal{H}_h^*(\boldsymbol{s}_h^3;\boldsymbol{z}_h). \tag{6}$$

As objective function in Problem (3) is convex in $\boldsymbol{x}_h$, the operating cost adopting $\pi_{heu}$ satisfies:

$$\mathcal{O}_h(\pi_{heu}, \boldsymbol{s}_h^3, \boldsymbol{z}_h) \le \alpha\mathcal{H}_h^*(\boldsymbol{s}_h^1;\boldsymbol{z}_h) + (1-\alpha)\mathcal{H}_h^*(\boldsymbol{s}_h^2;\boldsymbol{z}_h). \tag{7}$$

Combining (6) and (7), we verified Equation (4) and thus proved $\mathcal{H}_h^*(\boldsymbol{s}_h;\boldsymbol{z}_h)$ of Problem (3) is convex in $\boldsymbol{s}_h$. □

## Appendix B
## Proof of Lemma 2

*Proof.* For clarity, we use $\hat{\mathcal{G}}_h(\boldsymbol{x}_h)$ to denote $\hat{\mathcal{G}}_h(\boldsymbol{x}_h,\boldsymbol{s}_h;\boldsymbol{z}_h)$. To prove that Problem (2) is convex in $\boldsymbol{x}_h$, we will show that for any feasible $\boldsymbol{x}_h^1$ and $\boldsymbol{x}_h^2$, $\boldsymbol{x}_h^3 = \alpha\boldsymbol{x}_h^1 + (1-\alpha)\boldsymbol{x}_h^2$ is also feasible and satisfies

$$\alpha\hat{\mathcal{G}}_h(\boldsymbol{x}_h^1) + (1-\alpha)\hat{\mathcal{G}}_h(\boldsymbol{x}_h^2) \ge \hat{\mathcal{G}}_h(\boldsymbol{x}_h^3), \tag{8}$$

where $\alpha \in [0,1]$.

Let $\boldsymbol{s}_{h+1}^1$, $\boldsymbol{s}_{h+1}^2$, and $\boldsymbol{s}_{h+1}^3$ denote the system states at time $h+1$ following the decisions of $\boldsymbol{x}_h^1$, $\boldsymbol{x}_h^2$, and $\boldsymbol{x}_h^3$, respectively. When the decision is $\boldsymbol{x}_h^3$, the value of $\hat{\mathcal{G}}_h(\boldsymbol{x}_h^3)$ is

$$\hat{\mathcal{G}}_h(\boldsymbol{x}_h^3) = \mathcal{F}_h(\boldsymbol{x}_h^3;c_h) + \beta\mathbb{E}[\mathcal{H}_{h+1}^*(\boldsymbol{s}_{h+1}^3;\boldsymbol{z}_{h+1})], \tag{9}$$

Since $\mathcal{F}_h(\boldsymbol{x}_h;\boldsymbol{z}_h)$ is convex in $\boldsymbol{x}_h$, we have

$$\mathcal{F}_h(\boldsymbol{x}_h^3;c_h) \le \alpha\mathcal{F}_h(\boldsymbol{x}_h^1;c_h) + (1+\alpha)\mathcal{F}_h(\boldsymbol{x}_h^2;c_h). \tag{10}$$

Moreover, according to Equations (1a) and (1e), we have $\boldsymbol{s}_{h+1}^3 \ge \alpha\boldsymbol{s}_{h+1}^1 + (1-\alpha)\boldsymbol{s}_{h+1}^2$. According to Lemma 1, we have

$$\begin{aligned}
&\mathcal{H}_{h+1}^*(\boldsymbol{s}_{h+1}^3;\boldsymbol{z}_{h+1}) \\
\le&\mathcal{H}_{h+1}^*(\alpha\boldsymbol{s}_{h+1}^1 + (1-\alpha)\boldsymbol{s}_{h+1}^2;\boldsymbol{z}_{h+1}) \\
\le&\alpha\mathcal{H}_{h+1}^*(\boldsymbol{s}_{h+1}^1;\boldsymbol{z}_{h+1}) + (1-\alpha)\mathcal{H}_{h+1}^*(\boldsymbol{s}_{h+1}^2;\boldsymbol{z}_{h+1})
\end{aligned} \tag{11}$$

Combining (9)-(11), we verified Equation (8) and thus proved $\hat{\mathcal{G}}_h(\boldsymbol{x}_h)$ is convex in $\boldsymbol{x}_h$. □

[1] The subscripts in (1a)-(1i) are changed to $h+1$.

## APPENDIX C
## PROOF OF PROPOSITION 1

*Proof.*

$$\begin{aligned}\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out}) = &-\frac{Ac_h}{\eta(1-\epsilon)}[\epsilon T_h^{in}+(1-\epsilon)T_h^{out}]+\\ &\min_{T_{h+1}^{in}} \ \frac{Ac_h}{\eta(1-\epsilon)}T_{h+1}^{in}+\beta\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]\\ &\text{s.t. } T_{h+1}^{in}\geq \max[\underline{T},\epsilon T_h^{in}+(1-\epsilon)T_h^{out}],\\ &\quad T_{h+1}^{in}\leq \min[\overline{T},\epsilon T_h^{in}+(1-\epsilon)(T_h^{out}+\frac{\eta}{A}\bar{e})].\end{aligned} \tag{12}$$

Let $M(T_h^{in},c_h,T_h^{out})$ denote the optimal value of the minimization problem in (12), that is

$$\begin{aligned}&M(T_h^{in},c_h,T_h^{out}) =\\ &\min_{T_{h+1}^{in}} \ \frac{Ac_h}{\eta(1-\epsilon)}T_{h+1}^{in}+\beta\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]\\ &\text{s.t. } \max[\underline{T},\epsilon T_h^{in}+(1-\epsilon)T_h^{out}]\leq T_{h+1}^{in}\leq \overline{T}.\end{aligned} \tag{13}$$

When $\underline{T}\geq \epsilon T_h^{in}+(1-\epsilon)T_h^{out}$, i.e., $T_h^{out}\leq \frac{\underline{T}-\epsilon T_h^{in}}{1-\epsilon}$, $M(T_h^{in},c_h,T_h^{out})$ is independent of $T_h^{in}$. Hence, $\frac{\partial\mathbb{E}[\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out})]}{\partial T_h^{in}}=-\frac{\epsilon Ac_h}{\eta(1-\epsilon)}$ according to (12).

When $\underline{T}<\epsilon T_h^{in}+(1-\epsilon)T_h^{out}$, i.e., $T_h^{out}>\frac{\underline{T}-\epsilon T_h^{in}}{1-\epsilon}$, Problem (13) becomes

$$\begin{aligned}&M(T_h^{in},c_h,T_h^{out}) =\\ &\min_{T_{h+1}^{in}} \ \frac{Ac_h}{\eta(1-\epsilon)}T_{h+1}^{in}+\beta\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]\\ &\text{s.t. } \epsilon T_h^{in}+(1-\epsilon)T_h^{out}\leq T_{h+1}^{in}\leq \overline{T}.\end{aligned} \tag{14}$$

The optimal solution of (14) depends on the value of $\frac{\partial\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]}{\partial T_{h+1}^{in}}$, which is non-decreasing according to Lemma 1. For clear and concise, we denote $\left.\frac{\partial\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]}{\partial T_{h+1}^{in}}\right|_{\epsilon T_h^{in}+(1-\epsilon)T_h^{out}}$ and $\left.\frac{\partial\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]}{\partial T_{h+1}^{in}}\right|_{\overline{T}}$ by $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$, respectively.

- If $\mathcal{H}_2^*+\frac{Ac_h}{\beta\eta(1-\epsilon)}<0$, $\frac{\partial\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]}{\partial T_{h+1}^{in}}$ is always negative. The optimal solution and optimal value of (14) are $T_{h+1}^{in}=\overline{T}$ and $M(T_h^{in},c_h,T_h^{out})=\frac{Ac_h}{\eta(1-\epsilon)}\overline{T}+\beta\mathbb{E}[\mathcal{H}_{h+1}^*(\overline{T};c_{h+1},T_{h+1}^{out})]$, respectively. As $M(T_h^{in},c_h,T_h^{out})$ is independent of $T_h^{in}$, $\frac{\partial\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out})}{\partial T_h^{in}}=-\frac{Ac_h\epsilon}{\eta(1-\epsilon)}$.
- If $\mathcal{H}_1^*+\frac{Ac_h}{\beta\eta(1-\epsilon)}>0$, i.e., $c_h>-\frac{\beta\eta(1-\epsilon)\mathcal{H}_1^*}{A}=c_1$, then $T_{h+1}^{in}=\epsilon T_h^{in}+(1-\epsilon)T_h^{out}$. According to (12), we have $\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out}) = \beta\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]$. Thus, $\frac{\partial\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out})}{\partial T_h^{in}}=\beta\frac{\partial\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]}{\partial T_h^{in}}=\beta\mathcal{H}_1^*$.
- If $\mathcal{H}_1^*\leq-\frac{Ac_h}{\beta\eta(1-\epsilon)}\leq\mathcal{H}_2^*$, then the optimal $T_{h+1}^{in}$ satisfies $\frac{\partial\mathbb{E}[\mathcal{H}_{h+1}^*(T_{h+1}^{in};c_{h+1},T_{h+1}^{out})]}{\partial T_{h+1}^{in}}=-\frac{Ac_h}{\beta\eta(1-\epsilon)}$. As the optimal $T_{h+1}^{in}$ is independent of $T_h^{in}$, $M(T_h^{in},c_h,T_h^{out})$ is also independent of $T_h^{in}$. Hence, $\frac{\partial\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out})}{\partial T_h^{in}}=-\frac{Ac_h\epsilon}{\eta(1-\epsilon)}$.

With the above analysis, we conclude that

- When $T_h^{out}\leq\frac{\underline{T}-\epsilon T_h^{in}}{1-\epsilon}$, $\frac{\partial\mathbb{E}[\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out})]}{\partial T_h^{in}}=-\frac{\epsilon Ac_h}{\eta(1-\epsilon)}$.
- When $T_h^{out}>\frac{\underline{T}-\epsilon T_h^{in}}{1-\epsilon}$ and $\mathcal{H}_1^*+\frac{Ac_h}{\beta\eta(1-\epsilon)}\leq 0$ ($c_h\leq c_1$), $\frac{\partial\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out})}{\partial T_h^{in}}=\beta\mathcal{H}_1^*$.
- When $T_h^{out}>\frac{\underline{T}-\epsilon T_h^{in}}{1-\epsilon}$ and $\mathcal{H}_1^*+\frac{Ac_h}{\beta\eta(1-\epsilon)}>0$ ($c_h>c_1$), $\frac{\partial\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out})}{\partial T_h^{in}}=-\frac{Ac_h\epsilon}{\eta(1-\epsilon)}$.

Combine with the boundaries of $T_h^{out}$ and $c_h$, we have

$$\begin{aligned}&\frac{\partial\mathbb{E}[\mathcal{H}_h^*(T_h^{in};c_h,T_h^{out})]}{\partial T_h^{in}}\\ &=\int_{\underline{T}^{out}}^{T_\alpha^{out}}\int_{\underline{c}}^{\bar{c}}-\frac{Ac\epsilon}{\eta(1-\epsilon)}f(c)g(T^{out})\mathrm{d}c\mathrm{d}T^{out}\\ &+\int_{T_\alpha^{out}}^{\overline{T}^{out}}\int_{\underline{c}}^{c_1}-\frac{Ac\epsilon}{\eta(1-\epsilon)}f(c)g(T^{out})\mathrm{d}c\mathrm{d}T^{out}\\ &+\int_{T_\alpha^{out}}^{\overline{T}^{out}}\int_{c_1}^{\bar{c}}\beta\mathcal{H}_1^*f(c)g(T^{out})\mathrm{d}c\mathrm{d}T^{out}\\ &=-\frac{A\mu_c\epsilon}{\eta(1-\epsilon)}\\ &+\int_{T_\alpha^{out}}^{\overline{T}^{out}}\int_{c_1}^{\bar{c}}\big[\beta\mathcal{H}_1^*+\frac{Ac\epsilon}{\eta(1-\epsilon)}\big]f(c)g(T^{out})\mathrm{d}c\mathrm{d}T^{out}\\ &=-\frac{A\mu_c\epsilon}{\eta(1-\epsilon)}\\ &+\frac{A}{\eta(1-\epsilon)}\int_{T_\alpha^{out}}^{\overline{T}^{out}}\int_{c_1}^{\bar{c}}\big(c\epsilon-c_1\big)f(c)g(T^{out})\mathrm{d}c\mathrm{d}T^{out},\end{aligned} \tag{15}$$

where $T_\alpha^{out}=[\frac{\underline{T}-\epsilon T_h^{in}}{1-\epsilon}]_{\underline{T}^{out}}^{\overline{T}^{out}}$.

□

## APPENDIX D
## PROOF OF PROPOSITION 2

*Proof.*

$$\begin{aligned}&\mathcal{H}_h^*(E_h;c_h) =\\ &\min_{p_h} \ c_hp_h+\beta\mathbb{E}[\mathcal{H}_{h+1}^*(E_{h+1};c_{h+1})]\\ &\text{s.t. } (1e)-(1g).\end{aligned} \tag{16}$$

We prove $\frac{\partial\mathbb{E}[\mathcal{H}_h^*(E_h;c_h)]}{\partial E_h}$ is a non-increasing and piecewise constant function by mathematical induction initiating from the last time slot. In the mathematical induction, we also provide an iteration step to derive $\frac{\partial\mathbb{E}[\mathcal{H}_h^*(E_h;c_h)]}{\partial E_h}$.

When $h$ is the last time slot, i.e., $h=n$, the future cost will not be considered, i.e., $\mathbb{E}[\mathcal{H}_{n+1}^*(E_{n+1};c_h)]=0$. Then, the optimal solution for optimization (16) is $p_n^*=\eta_2\min[E_n,\bar{p}]$, i.e., the storage will run maximum discharge regardless of the amount of remaining storage and the price. Hence, the operating cost of the energy storage is $\mathcal{H}_n^*(E_n;c_n)=-\eta_2c_n\min[E_n,\bar{p}]$. Moreover,

$$\frac{\partial\mathbb{E}[\mathcal{H}_n^*(E_n;c_n)]}{\partial E_n}=\begin{cases}-\mu_c\eta_2, & E_n\in[0,\bar{p}]\\ 0, & \text{otherwise}\end{cases}, \tag{17}$$

which is a non-decreasing piecewise constant function.

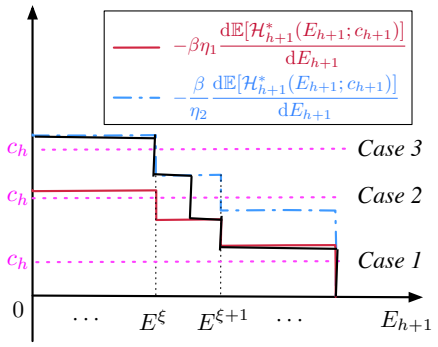


Fig. 1. The illustration of $\frac{\partial\zeta(p_h,E_h;c_h)}{\partial p_h}$.

Assume that $\frac{\partial\mathbb{E}[\mathcal{H}^*_h(E_{h+1};c_{h+1})]}{\partial E_{h+1}}$ is a non-decreasing piecewise constant function with respect to $E_{h+1}$ with $2\lambda_1+1$ pieces, with $c'_1\geq c'_2\geq\cdots c'_{2\lambda_2+1}$.

$$\frac{\partial\mathbb{E}[\mathcal{H}^*_{h+1}(E_{h+1};c_{h+1})]}{\partial E_{h+1}} =\begin{cases}-c'_0, & E_{h+1}\in[E^0,E^1]\\ -c'_1, & E_{h+1}\in[E^1,E^2]\\ \cdots & \\ -c'_{2\lambda_2}, & E_{h+1}\in[E^{2\lambda_2},E^{2\lambda_2+1}]\end{cases}, \tag{18}$$

where $E^\xi=\lfloor\frac{\xi}{2}\rfloor\bar{p}+(\xi-2\lfloor\frac{\xi}{2}\rfloor)\lambda_2,\forall\xi\in[0,2\lambda_1+1]$.

Let $\zeta(p_h,E_h;c_h)$ denote the objective in (16). The subderivative of $\zeta(p_h,E_h;c_h)$ is

$$\begin{aligned}\frac{\partial\zeta(p_h,E_h;c_h)}{\partial p_h}=&c_h+\beta\frac{\partial\mathbb{E}[\mathcal{H}^*_{h+1}(E_{h+1};c_{h+1})]}{\partial p_h}\\ =&c_h+\beta\bar{\eta}\frac{\partial\mathbb{E}[\mathcal{H}^*_{h+1}(E_{h+1};c_{h+1})]}{\partial E_{h+1}},\end{aligned} \tag{19}$$

where $\bar{\eta}=\begin{cases}\eta_1, & p_h>0\\ 1/\eta_2, & p_h<0\end{cases}$.

When $E_h\in[E^\xi,E^{\xi+1}]$, $-\beta\frac{\partial\mathbb{E}[\mathcal{H}^*_{h+1}(E_{h+1};c_{h+1})]}{\partial p_h}$ is described as the black line in Fig. 1. According to Fig. 1, the charging/discharging decision at time $h$ with $E_h\in[E^\xi,E^{\xi+1}]$ is

- *Case 1:* If
$$c_h+\beta\frac{\partial\mathbb{E}[\mathcal{H}^*_{h+1}(E_{h+1};c_{h+1})]}{\partial p_h}\Big|_{p_h=0}<0, \tag{20}$$
i.e., $c_h<\beta\eta_1c_\xi$, then $p^*_h>0$.
  - If $c_h<\beta\eta_1c_{\xi+2}$, $p^*_h=\frac{\bar{p}}{\eta_1}$. The operating cost of the energy storage is $\mathcal{H}^*_h(E_h;c_h)=\frac{\bar{p}}{\eta_1}+\beta\mathbb{E}[\mathcal{H}^*_{h+1}(E_h+\bar{p};c_{h+1})]$. The subderivative is $\frac{\partial\mathcal{H}^*_h(E_h;c_h)}{\partial E_h}=-\beta c_{\xi+2}$.
  - If $\beta\eta_1c_{\xi+2}\leq c_h<\beta\eta_1c_{\xi+1}$, $p^*_h=1/\eta_1(E^{\xi+2}-E_h)$. The operating cost of the energy storage is $\mathcal{H}^*_h(E_h;c_h)=c_h/\eta_1(E^{\xi+2}-E_h)+\beta\mathbb{E}[\mathcal{H}^*_{h+1}(E^{\xi+2};c_{h+1})]$. The subderivative is $\frac{\partial\mathcal{H}^*_h(E_h;c_h)}{\partial E_h}=-\frac{c_h}{\eta_1}$.
  - If $\beta\eta_1c_{\xi+1}\leq c_h<\beta\eta_1c_\xi$, $p^*_h=1/\eta_1(E^{\xi+1}-E_h)$. The operating cost of the energy storage is $\mathcal{H}^*_h(E_h;c_h)=c_h/\eta_1(E^{\xi+1}-E_h)+\beta\mathbb{E}[\mathcal{H}^*_{h+1}(E^{\xi+1};c_{h+1})]$. The subderivative is $\frac{\partial\mathcal{H}^*_h(E_h;c_h)}{\partial E_h}=-\frac{c_h}{\eta_1}$.
- *Case 2:* If $\beta\eta_1c_\xi\leq c_h\leq\beta/\eta_2c_\xi$, $p^*_h=0$. The operating cost of the energy storage is $\mathcal{H}^*_h(E_h;c_h)=\beta\mathbb{E}[\mathcal{H}^*_{h+1}(E_h;c_{h+1})]$. The subderivative is $\frac{\partial\mathcal{H}^*_h(E_h;c_h)}{\partial E_h}=-\beta c_{\xi-1}$.
- *Case 3:* If
$$c_h+\beta\frac{\partial\mathbb{E}[\mathcal{H}^*_{h+1}(E_{h+1};c_{h+1})]}{\partial p_h}\Big|_{p_h=0}>0. \tag{21}$$
i.e., $c_h>\frac{\beta c_\xi}{\eta_2}$, then $p^*_h<0$.
  - If $\frac{\beta c_\xi}{\eta_2}<c_h\leq\frac{\beta c_{\xi-1}}{\eta_2}$, $p^*_h=\eta_2(E^\xi-E_h)$. The operating cost of the energy storage is $\mathcal{H}^*_h(E_h;c_h)=\eta_2c_h(E^\xi-E_h)+\beta\mathbb{E}[\mathcal{H}^*_{h+1}(E^\xi;c_{h+1})]$. The subderivative is $\frac{\partial\mathcal{H}^*_h(E_h;c_h)}{\partial E_h}=-\eta_2c_h$.
  - If $\frac{\beta c_{\xi-1}}{\eta_2}<c_h\leq\frac{\beta c_{\xi-2}}{\eta_2}$, $p^*_h=\eta_2(E^{\xi-1}-E_h)$. The operating cost of the energy storage is $\mathcal{H}^*_h(E_h;c_h)=\eta_2c_h(E^{\xi-1}-E_h)+\beta\mathbb{E}[\mathcal{H}^*_{h+1}(E^{\xi-1};c_{h+1})]$. The subderivative is $\frac{\partial\mathcal{H}^*_h(E_h;c_h)}{\partial E_h}=-\eta_2c_h$.
  - If $c_h>\frac{\beta c_{\xi-2}}{\eta_2}$, $p^*_h=-\eta_2\bar{p}$. The operating cost of the energy storage is $\mathcal{H}^*_h(E_h;c_h)=-\eta_2c_h\bar{p}+\beta\mathbb{E}[\mathcal{H}^*_{h+1}(E_h-\bar{p};c_{h+1})]$. The subderivative is $\frac{\partial\mathcal{H}^*_h(E_h;c_h)}{\partial E_h}=-\beta c_{\xi-2}$.

Hence, $\frac{\partial\mathbb{E}[\mathcal{H}^*_h(E_h;c_h)]}{\partial E_h}$ is a non-decreasing and stepwise constant function. Moreover, $\frac{\partial\mathbb{E}[\mathcal{H}^*_h(E_h;c_h)]}{\partial E_h}$ with $E_h\in[E^\xi,E^{\xi+1}]$ can be derived by

$$\begin{aligned}&\frac{\partial\mathbb{E}[\mathcal{H}^*_h(E_h;c_h)]}{\partial E_h}\\ =&-\int_{\frac{\beta c_{\xi-2}}{\eta_2}}^{\bar{c}}\beta c_{\xi-2}f(c)\mathrm{d}c-\int_{\frac{\beta c_\xi}{\eta_2}}^{\frac{\beta c_{\xi-2}}{\eta_2}}\eta_2cf(c)\mathrm{d}c\\ &-\int_{\beta\eta_1c_\xi}^{\frac{\beta c_\xi}{\eta_2}}\beta c_{\xi-1}f(c)\mathrm{d}c-\int_{\beta\eta_1c_{\xi+2}}^{\beta\eta_1c_\xi}\frac{c}{\eta_1}f(c)\mathrm{d}c\\ &-\int_{\underline{c}}^{\eta_1c_{\xi+2}}\beta c_{\xi+2}f(c)\mathrm{d}c.\end{aligned} \tag{22}$$

□